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### ABSTRACT

This paper presents transient heat conduction analysis of composite nose cone subjected to aerodynamic heating by finite element technique in space domain and finite difference technique in time domain. An anisotropic rectangular ring element with four nodal circles, each having temperature as degree of freedom is developed. Application of finite element technique in space domain results in a set of first order coupled differential equations in time domain. These are solved by Crank-Nicholson finite difference scheme. Four test examples are used to validate the development of element and associated computer program. Practical application of the software developed is demonstrated by application to the problem of glass-epoxy nose cone subjected to a typical aerodynamic heat input.

### Introduction

Transient temperatures prediction by heat transfer analysis forms first phase of design of any aerospace structural component against aerodynamic heating. An integrated thermal and structural analysis via a finite element program is very much desirable for such applications. This has created lot of interest in finite element heat transfer analysis. Approaches to the formulation of element and system equations for finite element thermal analysis are parallel to those employed in structural analysis. Thus, a functional in terms of temperature is equivalent to 'potential energy' and has been termed as 'thermal potential'<sup>1</sup>. Counter parts of the 'mixed' and complementary energy principles also exist in thermal analysis. Complementary functional<sup>2</sup> is based upon heat flux as an independent variable, so that the mixed functional<sup>3</sup> employs both temperature and heat flux parameters. Today, interest in finite element thermal analysis has centered upon functional expressed in terms of temperature field. Some important contributions in this area are cited below.

Explicit formulation of first order triangular ring element is given by Brocci<sup>4</sup> and by Farhoomand<sup>5</sup>. The triangle and rectangle, both for planar and axisymmetric situations, are given in References<sup>6-10</sup>. Higher order rect-

angular element with 16 and 36 degrees of freedom are given by Rybicki and Hopper<sup>11</sup> and Skjolin and Cheung<sup>12</sup>. Isoparametric formulation concepts are used by Zienkiewicz and Parikh<sup>13</sup>. A more detailed review on the subject is given by Gallagher<sup>14</sup>. Thus it appears enough literature exists on finite element thermal analysis of isotropic bodies. However similar work on anisotropic composites is rather scarce. Padovan<sup>15-16</sup> gives finite element heat transfer analysis in anisotropic bodies.

In this work an anisotropic frustum of cone element with four nodal temperatures as degrees of freedom is described. This linear element is used for thermal analysis of laminated composite cone subjected to aerodynamic heating. A system of first order differential equations is obtained by finite element application in space. The solution of above set in time domain is obtained by employing Crank-Nicholson's finite difference scheme. Four test problems for which exact solutions are available, are solved to check the validity of formulation and computer programming. The temperatures in a laminated composite conical shell subjected to a typical aerodynamic heat input are determined by using this element.

### Finite Element Formulation

The geometry and the co-ordinate systems considered to represent the axisymmetric shells

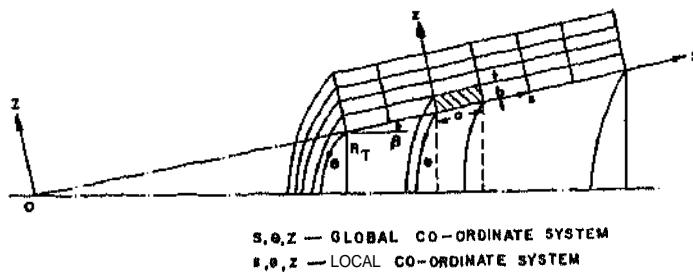


Fig. 1. Geometry and co-ordinate systems

of revolution are as shown in Fig. 1. The governing equation of conduction of heat in an axis-symmetric temperature field in an anisotropic shell of revolution is:

$$K_{ss} \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial T}{\partial s} \right) + K_{zz} \frac{\partial^2 T}{\partial z^2} + Q - \rho c \frac{\partial T}{\partial t} = 0 \quad (1)$$

$K_{ss}$  and  $K_{zz}$  are conductivity coefficients,  $T$  is temperature,  $Q$  is rate of heat generation,  $\rho$  is mass density,  $c$  is specific heat,  $t$  is time and  $s, z$  are coordinates.

The boundary conditions on the surface of shell of revolution are:

$$K_{zz} \frac{\partial T}{\partial z} = q_1 + H_1 (\psi_1 - T)$$

$$K_{ss} \frac{\partial T}{\partial s} = q_2 + H_2 (\psi_2 - T)$$

$$K_{zz} \frac{\partial T}{\partial z} = q_3 + H_3 (\psi_3 - T)$$

and

$$K_{ss} \frac{\partial T}{\partial s} = q_4 + H_4 (\psi_4 - T) \quad (2)$$

where  $q$  is flux positive outwards,  $H$  is surface heat transfer coefficient, and  $\psi$  is outside temperature.

The subscripts 1, 2, 3 and 4 denote the inner, bottom, outer and top surfaces of the shell of revolution. Equation (1) together with the boundary conditions (2), specifies the heat flow problem in a unique manner.

However, an alternative formulation based on calculus of variations is employed here.

The functional required for this is as follows:

$$\chi(T) = \int_{vol} \left\{ \frac{1}{2} K_{ss} T_s^2 + \frac{1}{2} K_{zz} T_z^2 - (Q - \rho c T_t) T \right\} dvol \quad (3)$$

in which  $T_t$  is constant.

The boundary conditions of the heat flow problem are accounted for by adding followir terms to the functional.

$$- \sum_{i=1}^4 \left\{ (q_i + H_i \psi_i) \int_{A_i} T dA \right\} + \sum_{i=1}^4 \left\{ H_i \int_{A_i} \frac{1}{2} T^2 dA \right\} \quad (4)$$

where  $A_1, A_2, A_3$  and  $A_4$  are surface areas corresponding to inner, bottom, outer and top surfaces.

Temperature is assumed to vary linearly within the element in both  $H$  and  $z$  directions, as shown below:

$$T = a_1 + a_2 s + a_3 z + a_4 sz \quad (5)$$

The temperature at any point within the element in both  $H$  and  $z$  directions, as shown below:

$$T = [N] \{T\}_e \quad (6)$$

where

$$N = [N_1, N_2, N_3, N_4]$$

and

$$[N]^T = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} = \begin{Bmatrix} 1 - s/a - z/b + sz/ab \\ s/a - sz/ab \\ sz/ab \\ z/b - sz/ab \end{Bmatrix} \quad (7)$$

The process of minimising the functional  $\chi$  is accomplished with respect to the nodal temperature  $H$ .

If  $\chi^e$  is functional associated with a given element then

$$\chi^e = \int_{vol} \left\{ \frac{1}{2} \{T_s, T_z\} \begin{bmatrix} K_{ss} & 0 \\ 0 & K_{zz} \end{bmatrix} \begin{Bmatrix} T_s \\ T_z \end{Bmatrix} - T(Q - \rho c T_t) \right\} dvol - \sum_{i=1}^4 (q_i + H_i \psi_i) \int_{A_i} T dA + \sum_{i=1}^4 \frac{1}{2} H_i^2 T^2 dA \quad (8)$$

using relation (6)

$$\begin{Bmatrix} T_s \\ T_z \end{Bmatrix} = [A] \{T\}_e$$

and

$$\{T_s, T_z\} = \{T\}_e^T [A]^T \quad (9)$$

where

$$[A] = \begin{bmatrix} N_{1,s} & N_{2,s} & N_{3,s} & N_{4,s} \\ N_{1,z} & N_{2,z} & N_{3,z} & N_{4,z} \end{bmatrix}$$

using relations (9) in relation (8)

$$\chi^e = \{T\}_e^T \left[ \int_{vol} \left\{ \frac{1}{2} [A]^T [K] [A] \{T\}_e - [N]^T (Q - \rho c [N] \{T\}_e) \right\} d vol \right.$$

$$\left. - \sum_{i=1}^4 (q_i + H_i \psi_i) \int_{A_i} [N]^T dA \right.$$

$$\left. + \sum_{i=1}^4 H_i \left( \int_{A_i} \frac{1}{2} [N]^T [N] dA \right) \{T\}_e \right]$$

Applying calculus of variation,

$$[g_b]_e \{T\}_e - Q \{f_b\}_e + \rho c [P_b]_e \{T\}_e$$

$$+ \sum_{i=1}^4 (q_i + H_i \psi_i) \{f_s\}_{ie} + \sum_{i=1}^4 H_i [g_s]_{ie} \{T\}_e = 0 \quad (11)$$

Equation (11) is rewritten as follows:

$$[g]_e \{T\}_e - \{f\}_e + [P]_e \{T\}_e = 0 \quad (12)$$

where

$$[g]_e = [g_b]_e + \sum_{i=1}^4 H_i [g_s]_{ie}$$

$$\{f\}_e = Q \{f_b\}_e + \sum_{i=1}^4 (q_i + H_i \psi_i) \{f_s\}_{ie}$$

Now for complete structure

$$[G] \{T\} - \{F\} + [P] \{T\} = 0 \quad (13)$$

where,

$[G]$  is the global conductivity matrix  
 $[P]$  is the global heat capacitance matrix  
 $\{T\}$  is the global nodal temperatures

$\{F\}$  is the global heat input vector  
 and  $\{T\}$  represents differentiation of  $\{T\}$  with respect to time.

### Finite Difference Scheme

Equations (13) are first order, linear coupled differential equations which are to be solved using initial conditions. These are solved by employing Crank-Nicholson finite difference scheme as this scheme never gives unstable oscillatory solution<sup>17</sup>.

According to this scheme,

$$\{T\}_t = \{T\}_{t-\Delta t} + \frac{\Delta t}{2} \left[ \{T\}_t + \{T\}_{t-\Delta t} \right] \quad (14)$$

now considering eqn. (13) at time 't' and at 't-Δt'

$$[G]_t \{T\}_t - \{F\}_t + [P] \{T\}_t = 0 \quad (15)$$

$$[G]_{t-\Delta t} \{T\}_{t-\Delta t} - \{F\}_{t-\Delta t} + [P] \{T\}_{t-\Delta t} = 0 \quad (16)$$

It is noted that  $[P]$  is independent of time 't'.

Using relations (14), (15) and (16)

$$([G]_t + \frac{2}{\Delta t} [P]) \{T\}_t = (-[G]_{t-\Delta t} + \frac{2}{\Delta t} [P]) \{T\}_{t-\Delta t} + \{F\}_t + \{F\}_{t-\Delta t} \quad (17)$$

$\{T\}_{t-\Delta t}$  is known from preceding solution (or as a vector of initial temperatures). Hence  $\{T\}_t$  is found from the solution of eqn. (17) which is in the form of

$$[\bar{G}] \{T\}_t = \{\bar{F}\} \quad (18)$$

where

$$[\bar{G}] = [G]_t + \frac{2}{\Delta t} [P]$$

and

$$\{\bar{F}\} = (-[G]_{t-\Delta t} + \frac{2}{\Delta t} [P]) \{T\}_{t-\Delta t} + \{F\}_t + \{F\}_{t-\Delta t}$$

### Numerical Evaluation

The finite element model and the computer programme are systematically evaluated for their performance in the analysis of axisymmetric steady state and transient temperature field problems by comparing the finite element solutions with the exact solutions of following example problems.

1. Steady state temperature distribution in a conical shell due to temperature difference between top and bottom edges (Fig. 2).

The exact solution for this problem, which can be obtained by directly integrating the governing differential equation, is

$$T = C_1 \log_e S + C_2$$

where  $C_1$  and  $C_2$  are constants of integration.

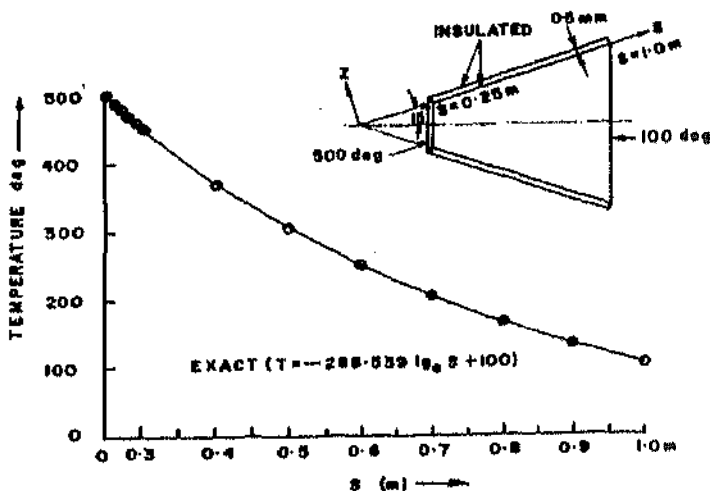


Fig. 2. Steady state temperature distribution in a conical shell due to the difference in temperatures between the edges.

2. Transient temperatures in a truncated conical shell with both top and bottom edges insulated under uniform heat input on the outer surface (Fig. 3).

A simple closed form solution for this problem is given in Ref. 18.

3. Transient temperature distribution in a conical shell with top edge at constant temperature and bottom edge insulated under uniform heat input on the outer surface (Fig. 4).

The exact solution for this problem in terms of Modified Bessel Functions is given in Ref. 18.

4. Transient temperature variation across the thickness in a truncated conical shell with both top and bottom edges insulated under uniform heat input on the outer surface (Fig. 5).

An analytical solution for this problem in Fourier series form is given in Ref. 19.

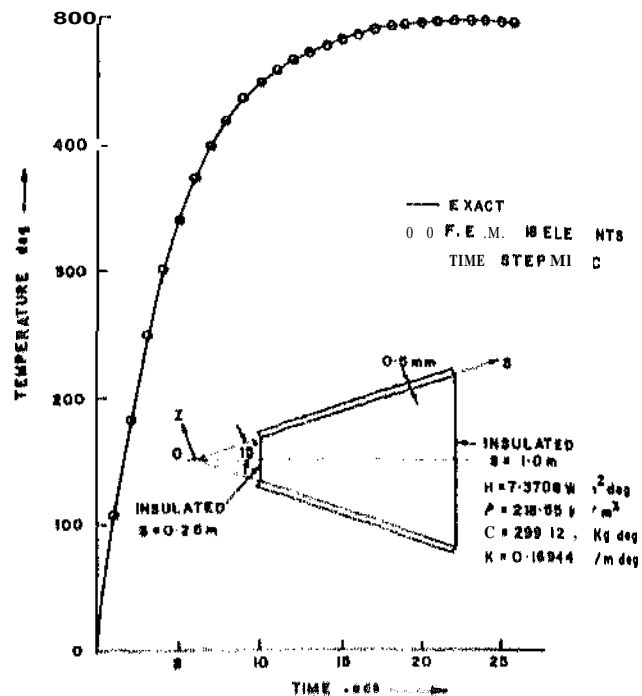


Fig. 3. Variation of temperature with time in a conical shell with insulated edges under uniform convective heat input.

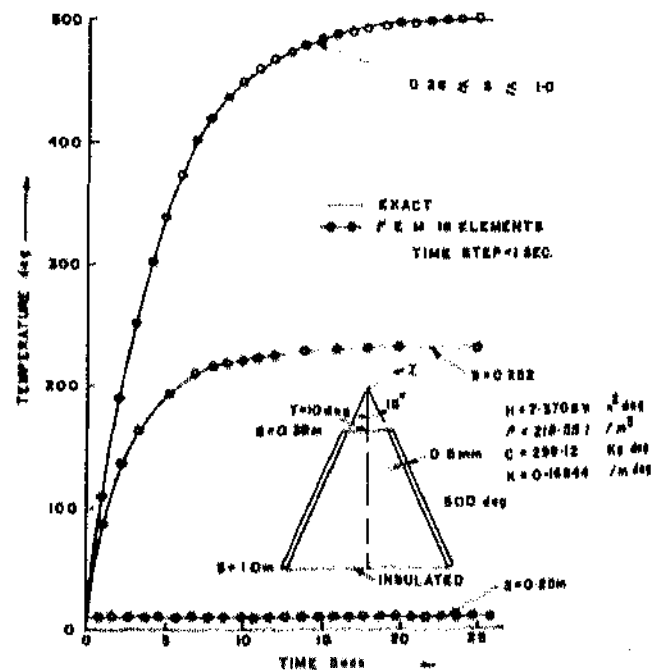


Fig. 4. Variation of temperature with time in a conical shell subjected to uniform convective heat input with top edge at constant temperature and bottom edge insulated.

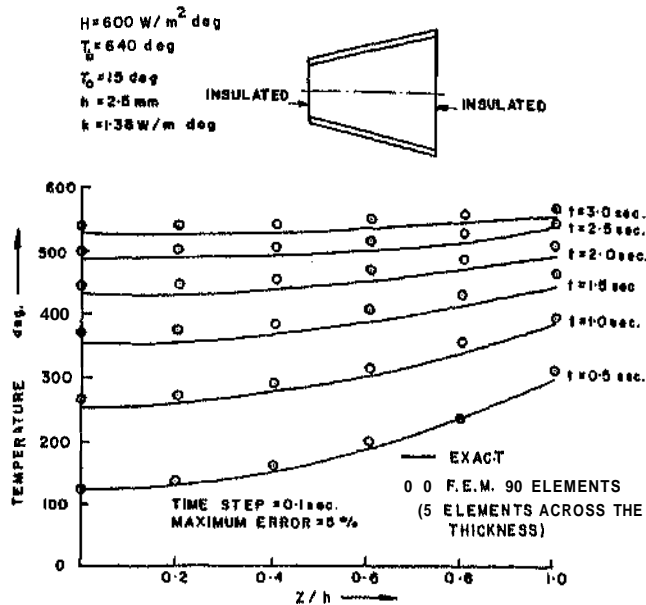


Fig.5. Variation of temperature across the thickness of a conical shell at various time intervals - insulated edges, uniform convective heat input,

In all the above test examples, the present method has yielded results in very close agreement with the ones obtained by analytical methods, as it can be seen from figures 2-5.

#### A Typical Practical Application

The method developed in the foregoing analysis was applied to a practical problem involving analysis of a laminated composite nose cone of a flight vehicle subjected to aerodynamic heat input which varies with time and space. The thermophysical properties, geometrical parameters and lamination parameters of the above mentioned shell are given in table 1. The convective heat transfer from boundary layer to the shell on the outer surface is considered whereas the inner surface is assumed to be insulated. The top surface of the shell is assumed to be at time dependent prescribed temperature. Further it is assumed that the prescribed temperature is the stagnation temperature. The bottom surface where, the shell is connected with rest of the structure is assumed to be insulated.

One of the problems faced in transient temperature prediction in nose cones of flight vehicles is the estimation of heat input due to aerodynamic heating which is quite a complex

Table 1. Data Used

Geometrical parameters of composite cone:

$\lambda = 1.0 \text{ m}$ ,  $(\delta = 15^\circ)$   
 Thickness = 2.5 mm  
 Lamination parameters of composite cone:  
 Stacking sequence  $\pm a$   
 $a = 60^\circ$ ,  $n = 10$   
 Thickness of each layer = .25 mm  
 Initial temperature of composite cone = 27 deg  
 Composite Material - Glass-Epoxy (60% glass by Volume)  
 Properties of composite material used:  
 Density =  $2.14 \times 10^3 \text{ kg/m}^3$   
 Specific heat = 937 J/Kg deg. Ref. 21  
 $K_L = 0.708 \text{ W/m deg}$   
 $K_T = 0.453 \text{ W/m deg}$ . Ref. 22.

phenomena. In the present application this estimate of heat input is done through an effective method described in Ref. 20. This method considers the flow past cone in all three regions - laminar, transitional and turbulent - and then an envelope of heat flux is obtained. This method yields maximum attainable heat transfer coefficients for various time intervals at different locations in the cone. Figure 6 indicates the variation of heat transfer coefficient along the generator of the cone at various time intervals. Table 2 shows the required boundary layer temperature  $T_b$  and stagnation temperature  $T_s$  at various time intervals.  $T_b$  and  $T_s$  are related by the following formula:

$$T_b - T_{\text{amb}} = 0.89 (T_s - T_{\text{amb}})$$

where  $T_{\text{amb}}$  is the ambient temperature.

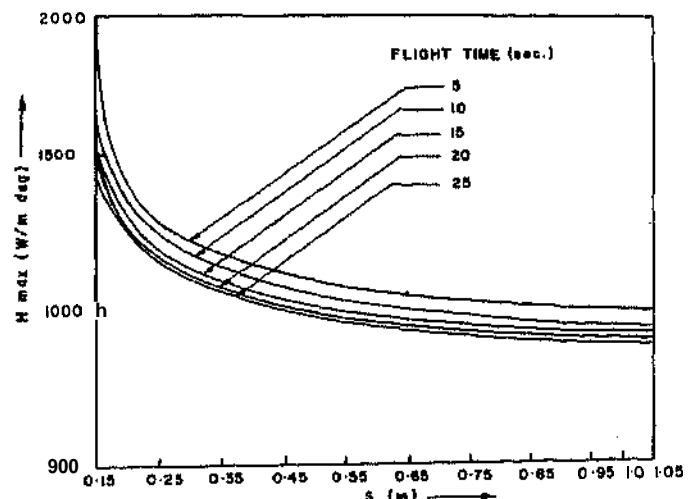


Fig.6. Variation of maximum heat transfer coefficient along the generator of cone at various time intervals - corresponding to a typical flight trajectory.

Table 2. Variation of  $T_b$  and  $T_s$  with time

t	$T_b$	$T_s$	t	$T_b$	$T_s$
0	15	15	14	148	171
1	36	39	15	150	176
2	57	62	16	164	189
3	78	86	17	178	202
4	99	110	18	189	210
5	120	133	19	196	218
6	122	136	20	203	226
7	124	139	21	211	236
8	128	144	22	220	246
9	139	150	23	229	256
10	138	156	24	239	266
11	141	159	25	249	276
12	144	163			
13	146	166			

Figure 7 shows the finite element grid used for this work. It is seen that the grid is finer at the top surface because of expected more severe temperature gradients there. Fig. 8 shows the variation of skin temperature along the generator at various time intervals. It appears that the temperature gradients are highly localised near the top surface but these can cause considerable amount of thermal stresses. Variation of skin temperature across the thickness is shown in Fig. 9.

### Conclusions

A linear rectangular anisotropic ring element is developed based on calculus of variation principle. This element and the associated computer program are evaluated by four well chosen test examples. It is demonstrated that this element can be successfully applied to solve the practical problem of transient temperatures prediction in laminated composite nose cone due to aerodynamic heating.

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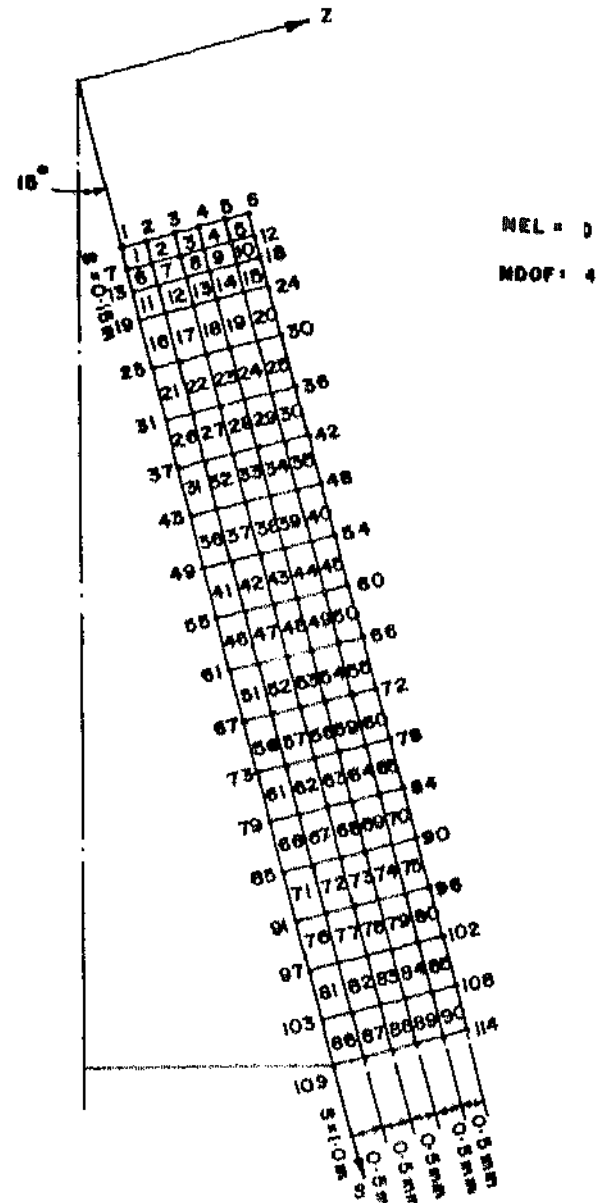


Fig. 7. Finite element idealization of laminated composite nose cone.

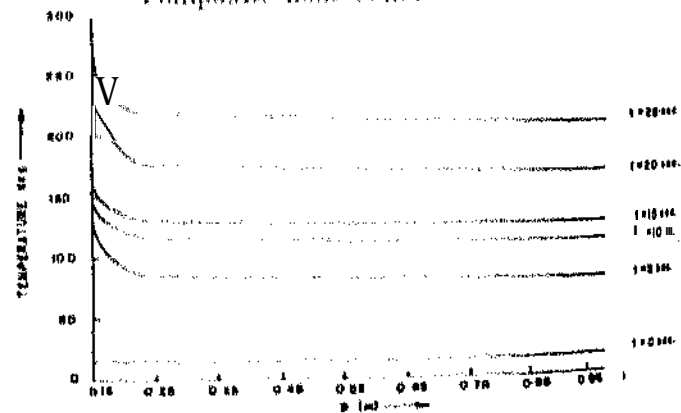


Fig. 8. Distribution of temperature along the generator of the nose cone at various time intervals.

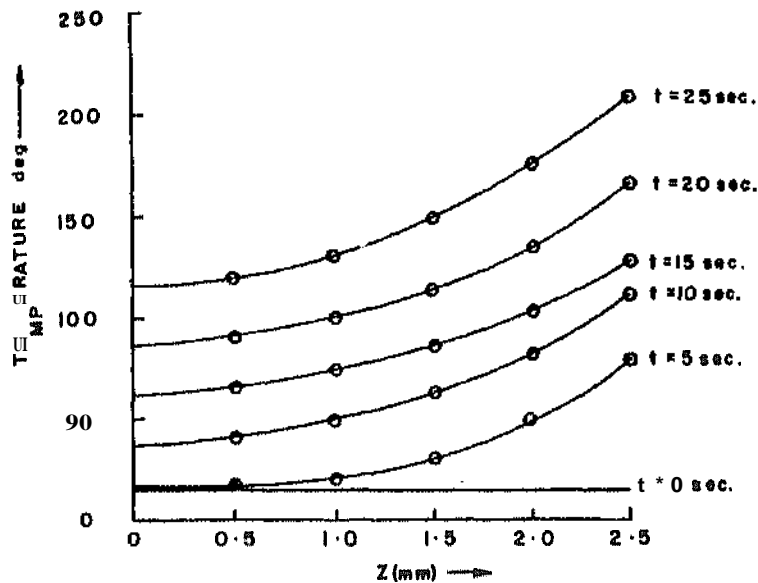


Fig.9. Temperature distribution across the thickness of the cone at various time intervals.

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